MAS224, Actuarial Mathematics: Select mortality

Consider two statements:-

(a) $\ _tq_x = \dfrac{s(x) - s(x+t)}{s(x)} = P(x < X \leq x + t | X > x)$; and

(b) $\ _tq_x = 1 - \ _tp_x = 1 - P(T(x) > t)$

In (a) we evaluate the probability that (x) will survive to age $x + t$ under the single hypothesis that the newborn has survived to age x; we disregard any additional information we might possess about chances of the further survival for (x). By doing this we implicitly assume that $T(x)$ is described by exactly the same survival function that describes X, as we assume that $P(T(x) > t) \equiv P(X > x + t | X > x) = \frac{s(x+t)}{s(x)}$. Hence we evaluate $\ _tq_x$ through the values of the survival function at ages x and $x + t$.

However, on the basis of additional information about (x) that we might have, we may decide that the use of $s(x)$ is no longer appropriate as it refers to the newborns and does not contain any particular information about (x).

Such additional information may be, for instance, that the life

- has just passed a medical examination for the purpose of life assurance;
- has just been treated for a serious illness or has just become disabled;
- has just taken out a life-annuity (higher chances of the survival compared to the general population, as a purchase of an annuity is worth to be considered only in good health, otherwise the purchase is likely to be a poor investment);
- has just entered the population from the outside world.

In (b) we evaluate $\ _tq_x$ directly from the probability that a life observed alive at age x will survive further t years; we may use whatever values of $\ _tp_x$ we decide are appropriate for the description of the mortality of (x) in the future.

As mentioned above, (a) and (b) are equivalent under the assumption that mortality rates depend only on age. Under this assumption the force of mortality acting on (x) is a function of $x + t$. In the previous lectures we assumed this implicitly and treated (a) and (b) as being equivalent.

Now we consider the situation when the force of mortality (or mortality rates) is a function of age **and** the time since a certain event known as "selection". We assume that individuals "(re)joined" the population after selection at age x will experience mortality that will be different over a certain time, known as the *select period*, from that of the general population.

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Notation:-

$[x]$	is used to denote the age of selection.
$([x]+k)$	denotes a person age $x+k$ that (re)joined the population after selection at age x.
$T([x]+k)$	is the future life time (the time-until-death) for a person age $x+k$ who was selected at age x.
$p_{[x]}$	is the probability to survive at least one further year after selection at age x (i.e. $[x]$ will attain age $x+1$), $p_{[x]} = P(\,T([x]) > 1\,)$.
$p_{[x]+k}$	is the probability for a person age $x+k$, who was selected at age x, to survive to age $x+k+1$, $p_{[x]+k} = P(\,T([x]+k) > 1\,)$.
${}_tp_{[x]+k}$	is the probability for a person aged $x+k$, who was selected at age x, to survive to age $x+t+k$, ${}_tp_{[x]+k} = P(\,T([x]+k) > t\,)$.
${}_tq_{[x]+k}$	is the probability that a person selected at age x and being currently observed alive at age $x+k$ will die within t years, ${}_tq_{[x]+k} = P(\,T([x]+k) \le t\,)$.
$\mathring{e}_{[x]+k}$	is the complete expectation of the further life for a person selected at age x and being currently observed alive at age $x+k$.

Denote by $l_{[x]+k}$ the expected number of survivors to age $x+k$ in a group of $l_{[x]}$ individuals who (re)joined the population at age x (the same sort of selection for all members of the group). Then

$$
\begin{aligned}
{}_tp_{[x]+k} &= \frac{l_{[x]+k+t}}{l_{[x]+k}}, \quad t>0, k=0,1,2,...\\
{}_tq_{[x]+k} &= 1-{}_tp_{[x]+k} = \frac{l_{[x]+k}-l_{[x]+k+t}}{l_{[x]+k}}, \quad t>0, k=0,1,2,...,
\end{aligned}
$$

i.e. the life-table functions are linked in the normal way over the range of select values.

Beyond the select period the mortality rates of selected lives are assumed to reverse to the mortality rates of the general population. Therefore,

$$p_{[x]+n+k} = p_{x+n+k}, \quad \text{for all } k \ge 0, \quad n \text{ being the duration of select period.}$$

More generally, in all formulae we will simply use $x+n+k$ instead of $[x]+n+k$, provided the duration of the select period is n years. There is no difference in terms of mortality between $(x+n+k)$ and $([x]+n+k)$.

Worked Example 1.
Find $q_{[52]}$, q_{52}, $q_{[52]+1}$, ${}_2q_{[52]+1}$ on the basis of the life assurance table A1967-70.

Solution.
This table has select period of 2 years, $n = 2$.

$$q_{[52]} = \frac{l_{[52]} - l_{[52]+1}}{l_{[52]}} = \frac{32188.740 - 32077.958}{32188.740} = 0.0034 \qquad \text{(4 d.p.)}$$

$$q_{52} = \frac{l_{52} - l_{53}}{l_{52}} = \frac{32338.568 - 32143.546}{32338.568} = 0.0060 \qquad \text{(4 d.p.)}$$

$$\begin{aligned} q_{[52]+1} &= \frac{l_{[52]+1} - l_{[52]+2}}{l_{[52]+1}} \\ &= \frac{l_{[52]+1} - l_{52+2}}{l_{[52]+1}} = \frac{l_{[52]+1} - l_{54}}{l_{[52]+1}} = \frac{32077.958 - 31926.430}{32077.958} = 0.0047 \qquad \text{(4 d.p.)} \end{aligned}$$

$$\begin{aligned} {}_2q_{[52]+1} &= \frac{l_{[52]+1} - l_{[52]+1+2}}{l_{[52]+1}} \\ &= \frac{l_{[52]+1} - l_{52+3}}{l_{[52]+1}} = \frac{l_{[52]+1} - l_{55}}{l_{[52]+1}} = \frac{32077.958 - 31685.203}{32077.958} = 0.0122 \qquad \text{(4 d.p.)} \end{aligned}$$

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Worked Example 2.

Assume mortality rates of the English Life Table No. 12 – Males. A life assurance is being taken out by a 20-year old whose chances of survival during the following two years are known to be higher than those of the general population: $p_{[20]} = \frac{1}{2}(1 + p_{20})$ and $p_{[21]} = \frac{1}{2}(1 + p_{21})$. Find the complete expectation of life for this person at age 20.

Solution.

Need to find $\mathring{e}_{[20]}$.

Age of selection is 20 years, hence [20]. The duration of selection period is 2 years, hence $[20] + 2 + k = 22 + k$ in all formulae. This means that, in the range of ages from 22 upwards, we can use the ELT-12-Males values of the life-table functions. So can use $p_{22}, p_{23}, ..., \mathring{e}_{22}$. We can also use $l_{22}, l_{23}, ...,$ provided we extrapolate the values of $l_{[20]+1}$ and $l_{[20]}$ from l_{22} with the help of $p_{[20]}$ and $p_{[21]}$.

$$l_{[20]+1} = \frac{l_{22}}{p_{[20]+1}}, \text{ as } p_{[20]+1} = \frac{l_{[20]+2}}{l_{[20]+1}} = \frac{l_{22}}{l_{[20]+1}}.$$

$$\therefore l_{[20]+1} = \frac{l_{22}}{\frac{1}{2}(1 + p_{21})} = \frac{96065}{\frac{1}{2}(1 + 0.99882)} = 96122$$

Similarly, $l_{[20]} = \frac{l_{[20]+1}}{p_{[20]}} = \frac{l_{[20]+1}}{\frac{1}{2}(1 + p_{20})} = \frac{96122}{\frac{1}{2}(1 + 0.99881)} = 96179.$

Therefore

$$\begin{aligned}
\mathring{e}_{[20]} \simeq \frac{1}{2} + \mathsf{e}_{[20]} &= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{l_{[20]+k}}{l_{[20]}} \\
&= \frac{1}{2} + \frac{1}{l_{[20]}} \left(l_{[20]+1} + l_{[20]+2} + l_{[20]+3} + \ldots \right) \\
&= \frac{1}{2} + \frac{1}{l_{[20]}} \left(l_{[20]+1} + l_{22} + l_{23} + \ldots \right) \\
&= \frac{1}{2} + \frac{1}{l_{[20]}} \left[l_{[20]+1} + \frac{l_{21}}{l_{21}} \left(l_{22} + l_{23} + \ldots \right) \right] \\
&= \frac{1}{2} + \frac{1}{l_{[20]}} \left(l_{[20]+1} + l_{21} \mathsf{e}_{21} \right) \\
&= \frac{1}{2} + \frac{l_{[20]+1}}{l_{[20]}} + \frac{l_{21}}{l_{[20]}} \mathsf{e}_{21} \\
&\simeq \frac{1}{2} + p_{[20]} + \frac{l_{21}}{l_{[20]}} \left(\mathring{e}_{21} - \frac{1}{2} \right) \\
&= 0.5 + \frac{1}{2}\Big(1 + 0.99881\Big) + \frac{96178}{96179}\Big(49.63 - 0.5\Big) = 50.63
\end{aligned}$$